CHAPTER 11 -- VIBRATORY MOTION

11.1) *A = .5 meters*; *T = .3 sec/cycle*; and *m = 1.2 kg*.

a.) Oscillatory frequency:

$$\begin{aligned}\nu &= 1/T \\ &= 1/(.3 \text{ sec/cycle}) \\ &= 3.33 \text{ cycles/sec} \quad (\text{or } 3.33 \text{ hz}).\end{aligned}$$

b.) Angular frequency (the number of *radians per second* the motion sweeps through):

$$\begin{aligned}\omega &= 2\pi\nu \\ &= 2(3.14)(3.33 \text{ hz}) \\ &= 20.94 \text{ rad/sec.}\end{aligned}$$

c.) Spring constant:

$$\omega = (k/m)^{1/2}$$
$$\Rightarrow \quad \begin{aligned}k &= \omega^2 m \\ &= (20.94 \text{ rad/sec})^2(1.2 \text{ kg}) \\ &= 526.4 \text{ nt/m.}\end{aligned}$$

d.) Maximum velocity (occurs where acceleration is zero, or while mass passes through equilibrium position):

$$\begin{aligned}v_{max} &= \omega A \\ &= (20.94 \text{ rad/sec})(.5 \text{ m}) \\ &= 10.47 \text{ m/s.}\end{aligned}$$

e.) Magnitude of the maximum acceleration (occurs where force is greatest, or at extremes of motion):

$$\begin{aligned}a_{max} &= \omega^2 A \\ &= (20.94 \text{ rad/sec})^2(.5 \text{ m}) \\ &= 219.2 \text{ m/s}^2.\end{aligned}$$

f.) Energy in system:

$$E = (1/2)kA^2$$
$$= .5\ (526.4\ nt/m)(.5\ m)^2$$
$$= 65.8\ joules.$$

11.2) *m = .25 kg; k = 500 nt/m;* v_{max} *= 3 m/s.*

a.) Angular frequency:

$$\omega = (k/m)^{1/2}$$
$$= [(500\ nt/m)/(.25\ kg)]^{1/2}$$
$$= 44.72\ rad/sec.$$

b.) Frequency:

$$\omega = 2\pi\nu$$
$$\Rightarrow \quad \nu = \omega/2\pi$$
$$= (44.72\ rad/sec)/2\pi$$
$$= 7.12\ hz.$$

c.) Period:

$$T = 1/\nu$$
$$= 1/(7.12\ hz)$$
$$= .14\ sec/cycle.$$

d.) Amplitude:

$$v_{max} = \omega A$$
$$\Rightarrow \quad A = v_{max}/\omega$$
$$= (3\ m/s)/(44.72\ rad/sec)$$
$$= .067\ m.$$

e.) Total energy:

$$E_t = (1/2)kA^2$$
$$= .5(500\ nt/m)(.067\ m)^2$$
$$= 1.125\ joules.$$

f.) Magnitude of maximum force (this will be applied when the acceleration is maximum--i.e., at the extremes):

$$a_{max} = \omega^2 A$$
$$= (44.72 \text{ rad/sec})^2(.067 \text{ m})$$
$$= 134 \text{ m/s}^2.$$

N.S.L. $\Rightarrow$

$$F_{max.} = ma_{max}$$
$$= (.25 \text{ kg})(134 \text{ m/s}^2)$$
$$= 33.5 \text{ nts.}$$

11.3) The most general *position as a function of time* expression is:

$$x = A \sin(\omega t \pm \phi) \quad \text{or} \quad x = A \sin(2\pi \nu t \pm \phi).$$

In our case, we know that $x = .7 \sin(14t - .35)$. Matching the appropriate variables leads to:

a.) Amplitude: A = .7 meters (by inspection).

b.) Angular frequency: ω = 14 rad/sec (by inspection).

c.) Frequency:

$$\omega = 2\pi\nu$$
$$\Rightarrow \quad \nu = \omega/2\pi$$
$$= (14 \text{ rad/sec})/2\pi$$
$$= 2.23 \text{ hz.}$$

d.) Position at $t = 3$ *seconds*:

$$x = .7 \sin(14t - .35)$$
$$= .7 \sin(14(3 \text{ sec}) - .35)$$
$$= -.51 \text{ m.}$$

e.) Position at $t = 3.4$ *seconds*:

$$x = .7 \sin(14t - .35)$$
$$= .7 \sin(14(3.4 \text{ sec}) - .35)$$
$$= -.088 \text{ m.}$$

f.) Velocity at $t = 0$:

$$\begin{aligned} v &= \omega A \cos(\omega t + \phi) \\ &= (14 \text{ rad/sec})(.7 \text{ m}) \cos(14t - .35) \\ &= (14 \text{ rad/sec})(.7 \text{ m}) \cos(14(0) - .35) \\ &= 9.2 \text{ m/s}. \end{aligned}$$

g.) Acceleration at $t = 0$:

$$\begin{aligned} a &= -\,\omega^2 A \sin(\omega t + \phi) \\ &= -(14 \text{ rad/sec})^2(.7 \text{ m}) \sin(14t - .35) \\ &= -(14 \text{ rad/sec})^2(.7 \text{ m}) \sin(14(0) - .35) \\ &= 47 \text{ m/s}^2. \end{aligned}$$

11.4) The period of oscillation T is related to the motion's frequency ν by the relationship $\nu = 1/T$. The frequency ν is related to the *angular frequency* ω by the relationship $\omega = 2\pi\nu$. In simple harmonic motion, the angular frequency is a function of the pendulum arm length L and the acceleration of gravity g. That is:

$$\begin{aligned} \omega &= (g/L)^{1/2} \\ \Rightarrow \quad g &= \omega^2 L \\ &= (2\pi\nu)^2 L \\ &= (2\pi/T)^2 L \\ &= (2\pi/(2 \text{ sec}))^2 (1.75 \text{ m}) \\ &= 17.27 \text{ m/s}^2. \end{aligned}$$

Notice that the mass has nothing to do with anything here (just as the mass would have nothing to do with the rate at which velocity changes as a body falls near the planet's surface).

11.5)

a.) The equation $\alpha + (12g/7L)\ \theta = 0$ is of the form "acceleration plus constant-times-displacement equals zero" (even though the acceleration and displacement terms are angular ones). That means the motion is, by definition, *simple harmonic* in nature.

b.) For *simple harmonic motion*, we know that the *square of the angular frequency* of the motion is equal to the constant in front of the *displacement* term. In this case:

$$\omega^2 = 12g/7L$$
$$\Rightarrow \quad \omega = (12g/7L)^{1/2}.$$

We also know that $w = 2\pi\nu$, or:

$$\begin{aligned}\nu &= \omega/2\pi \\ &= [(12g)/(7L)]^{1/2}/(2\pi) \\ &= (1.7g/L)^{1/2}/(2\pi) \\ &= [(1.7)(9.8\ m/s^2)/(1.3\ m)]^{1/2}/[(2(3.14)] \\ &= .57\ \text{cycles/second.}\end{aligned}$$

11.6) Note that when the 3 kg mass is attached to the spring, gravity acting on the mass applies a force such that the spring elongates .7 meters. Also, at $t = 0$, $y = -.15$ *meters* moving *away from* equilibrium:

a.) Spring constant: The spring constant tells you how much force F is required to elongate the spring. In this case, GRAVITY is used to elongate the spring a distance d, or:

$$\begin{aligned}k &= (mg)/d \\ &= (3\ kg)(9.8\ m/s^2)/(.7\ m) \\ &= 42\ nt/m.\end{aligned}$$

b.) Angular frequency:

$$\begin{aligned}\omega &= (k/m)^{1/2} \\ &= [(42\ nt/m)/(3\ kg)]^{1/2} \\ &= 3.74\ rad/sec.\end{aligned}$$

c.) Amplitude: $A = .4$ *meters* (stated in problem: "once at equilibrium, the system is displaced an additional .4 meters and released").

d.) Frequency:

$$\omega = 2\pi\nu$$
$$\begin{aligned}\Rightarrow \quad \nu &= \omega/2\pi \\ &= (3.74\ rad/sec)/2\pi \\ &= .6\ hz.\end{aligned}$$

e.) Period:

$$T = 1/\nu$$
$$= 1/(.6 \text{ cycles/sec})$$
$$= 1.67 \text{ sec/cycle.}$$

f.) Total energy:

$$E_t = (1/2)kA^2$$
$$= .5(42 \text{ nt/m})(.4 \text{ m})^2$$
$$= 3.36 \text{ joules.}$$

g.) Maximum velocity:

$$v_{max} = \omega A$$
$$= (3.74 \text{ rad/sec})(.4 \text{ m})$$
$$= 1.5 \text{ m/s.}$$

h.) The velocity is maximum at equilibrium.

i.) Maximum acceleration:

$$a_{max} = \omega^2 A$$
$$= (3.74 \text{ rad/sec})^2(.4 \text{ m})$$
$$= 5.6 \text{ m/s}^2.$$

j.) The acceleration is maximum at either extreme (i.e., at $x = \pm A$).

k.) To determine the general algebraic expression (using y as the position variable), we start with the standard form:

$$y = A \sin(\omega t + \phi) \quad \text{or} \quad y = A \sin(2\pi \nu t + \phi).$$

Knowing the *amplitude* and the *angular frequency*, we can immediately write:

$$y = .4 \sin(3.74t + \phi).$$

The only variable we need to determine anew is the phase shift ϕ. We know that at $t = 0$, $y = -.15$ moving away from equilibrium. The sine wave shown on the next page highlights this situation. To determine the phase shift ϕ:

We know that in general,

$$y = .4 \sin (3.74t + \phi).$$

Substituting in $t = 0$ and $y = -.15$, we can write:

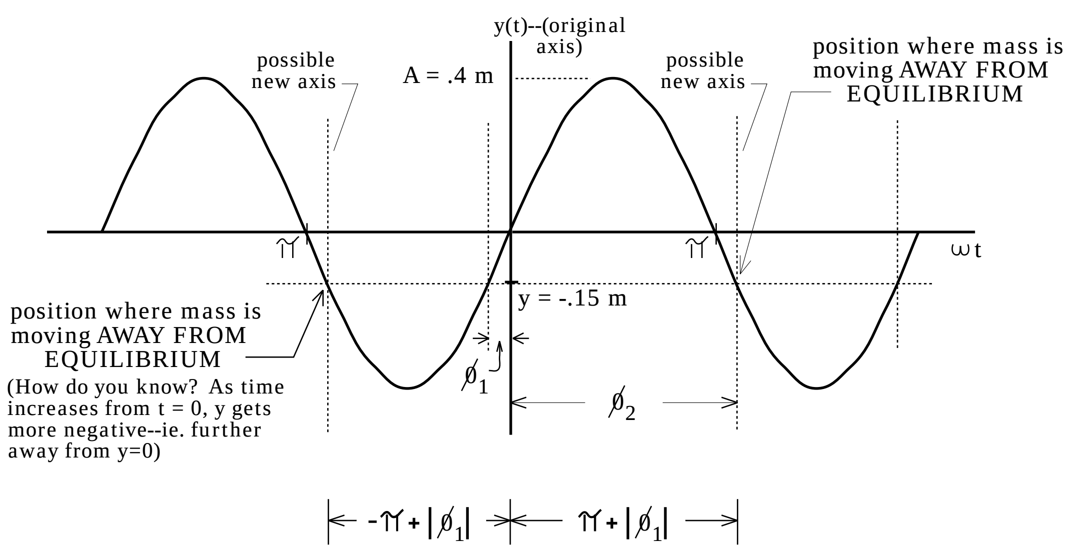
$$-.15 = .4 \sin (3.74(0) + \phi)$$
$$-.15 = .4 \sin (\phi).$$

Solving for ϕ yields:

$$\phi = \sin^{-1} [(-.15)/(.4)]$$
$$= -.384 \text{ rad.}$$

The question is, "What does that mean?" That is, there are two angles that could possibly satisfy the math in this situation, one in the fourth quadrant (ϕ_1 in the sketch below) and one in the third quadrant (ϕ_2). We have been given a value for the fourth quadrant angle (that is, $\phi_1 = -.384$ *radians*). We can find the other using math and a bit of logic.

To tell which of the two we really want, look at the sketch below (both possible axes are shown).



From observation, it is evident that the object is moving *away from* equilibrium when in the *third* quadrant. There is no formula to get the angle required to move the axis to the appropriate third-quadrant position on the sine wave (we can move left or right to get there--either will do), so we will have to use our heads.

Doing so, the sketch allows us to see that ϕ_2 is equal to the addition of the *magnitude* of ϕ_1 and $-\pi$ (i.e., $-3.14 + .384 = -2.76$ *rad*). Another possibility would be to add the magnitude of ϕ_1 to $+\pi$ (i.e., $3.14 + .384 = 3.53$ *rad*). Both cases are shown in the figure; both cases designate an appropriate position for the new axis.

Bottom line: The general algebraic expression that defines the *position* of the body as a function of *time* is:

$$y = .4 \sin (3.74t - 2.76)$$

or

$$y = .4 \sin (3.74t + 3.53).$$

Either is acceptable.

11.7) Although this probably appears to be a completely off-the-wall problem, it is characteristic of a class of problems that have a common hallmark. Specifically, they all give information about the force acting on a body moving in periodic motion (or they give enough information for you to derive a force relationship--in this case, we derived the required expression in the previous chapter), and they always ask something about the *period* of the motion.

a.) If we can determine Jack's period, we will have the period of the satellite (the two have to be the same if the satellite is going to take a picture of Jack head every time it appears out the top (at the bottom, it will be his feet that will appear). The key to finding Jack's period is found in the gravitational force that keeps him oscillating between poles.

Assuming the motion is in the *y* direction and leaving the sign of the acceleration embedded in the a_y term, we can write:

$\underline{\sum F_y:}$

$$-\left[\frac{Gm_e m_J}{r_e^{\,3}}\right]y = m_J a_y .$$

Rearranging, this implies that:

$$a_y + \left[\frac{Gm_e}{r_e^{\,3}}\right]y = 0 .$$

This is the characteristic equation for *simple harmonic motion*. Knowing that, we also know that the *angular frequency* of Jack's oscillatory motion must be equal to the *square root of the constant in front of the displacement term*. Using our relationships between *angular frequency*, *frequency*, and *period*, we can write:

$$\omega = (Gm_e/r_e^3)^{1/2}.$$

As

$$\omega = 2\pi\nu$$

and

$$T = 1/\nu,$$

we can write:

$$T = 2\pi/(Gm_e/r_e^3)^{1/2}.$$

b.) As the note in the problem said, this really isn't a vibratory motion problem--it's more of a gravitation problem in review. No matter. The practice will do you good.

Knowing the period of motion, we can generate an expression for the velocity of the mass in terms of its radius. That is:

$$\begin{aligned} v_s &= (\text{circumference of orbit})/T \\ &= [2\pi r_{satellite}]/[2\pi/(Gm_e/r_e^3)^{1/2}] \\ &= (Gm_e/r_e^3)^{1/2} r_s. \end{aligned}$$

Knowing the velocity, we can use N.S.L. and the fact that the satellite's motion is centripetal (we'll assume a circular path for simplicity). Using that approach, we get:

$\underline{\sum F_c:}$

$$Gm_e m_s/r_s^2 = m_s(v_s^2/r_s)$$
$$\Rightarrow \quad r_s = (Gm_e/v_s^2).$$

Plugging this back into our velocity expression above, we get:

$$\begin{aligned} v_s &= (Gm_e/r_e^3)^{1/2} r_s \\ &= (Gm_e/r_e^3)^{1/2}(Gm_e/v_s^2) \\ \Rightarrow \quad v_s &= [(Gm_e/r_e^3)^{1/2} Gm_e]^{1/3} \\ &= (Gm_e/r_e)^{1/2}. \end{aligned}$$

We derived an expression for r_s in terms of v_s. Substituting v_s into that expression yields:

$$\begin{aligned} r_s &= (Gm_e/v_s^{2}) \\ &= Gm_e/[(Gm_e/r_e)^{1/2}]^2 \\ &= r_e. \end{aligned}$$

Amazing! The satellite would have to skim the earth's surface to have the velocity associated with the correct period.

Wasn't that fun!